SOME MATHEMATICAL PROBLEMS AND THEIR SOLUTIONS FOR THE OSCILLATING SYSTEMS WITH LIQUID DAMPERS (Survey)

Abstract. The mathematical problem of an oscillating system with liquid dampers is considered, such as finding the order of the fractional derivative of a subordinate term based on the given statistical data from practice, constructing a solution of the corresponding system with nonseparated boundary conditions, including for large values of the head mass, finding asymptotic solutions on the first approximations, and constructing optimal controllers to stabilize the system around the corresponding program trajectories and controls.

Keywords: oscillating systems, liquid damper, fractional derivative, asymptotic representation–n, optimal control, regulator, boundary conditions.

1. **Introduction.** The differential equations [22,28, 29, 30,38] of the classical oscillating system

$$m\ddot{y}(t)+a\dot{y}(t)+by(t)=f(t) \qquad (1.1)$$

play an important role in solving many problems of control [32], optimization [20,49,58], oil production of sucker-rod pumping units [17-19,64], etc., where *m* is the mass of the head, *a* and *b* are given real numbers with concrete physical significance, *f*(*t*) is an external disturbance, a continuous real-valued function.

If the head of the oscillating system moves inside the Newtonian fluid, then the control (1.1) ceases to describe the resulting process exactly and (1.1) is reduced to the next control of the fractional derivative in the subordinate term [1,3,5,8,11,13,14]

$$m\ddot{y}(t)+aD^{\alpha}y(t)+by(t)=f(t),$$

$\alpha \in (0,1)\cup(1,2)$.

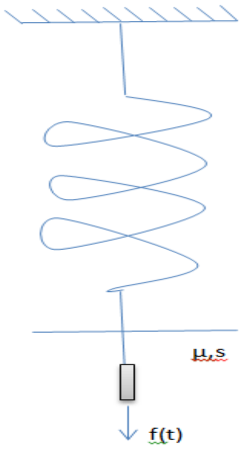

Fig.1

For equation (1.2), various problems can be considered:

1. Determining the order, which is the fractional derivative of the subordinate term, using statistical data from practice (e.g., in oil production with a sucker rod pump unit, statistical data can be taken from the volume of the flow rate at different points in time).
2. Solution of a non-local boundary value problem for equation (1.2).
3. For sufficiently large values of *m*, finding the asymptotic representation by the first approximation . Note that the results of (2), (3) can be successfully used to find program trajectories and controls for oil delivery by a sucker-rod pumping unit [23].
4. Finding the optimal controllers to stabilize the motion of the head mass m around the corresponding program trajectories and controls. Here the controllers can be designed using time-frequency methods.

In our opinion, the second method is more acceptable for developing efficient computational methods.

Note that the solution of the problem given in items 1-4 is considered in the papers [21,22,40-43,68,78] for simple cases when m, a, b are constant real numbers. In the case of the motion of the head of mass m in the Newtonian fluid (in the case of oil production, the motion of the plunger in the fluid) is a function of *t* and as a consequence, after division by *m*, the coefficients of the obtained two terms depend on *t*. Therefore, the above methods are complicated and it is necessary to take into account the periodic problems [26,41] of choice of program trajectories and control, as well as their optimal stabilization [44]. In this case, the determination of the fractional derivative becomes rather complicated and requires the discretization of the corresponding equation (1.2) [43]. Furthermore, the problem of selecting program trajectories, controls and optimal stabilization is posed for the periodic system where *m* and *y*operate according to this principle. This approach is also acceptable for the fact that the motion of the plunger in *m* and *y* is described by either differential or finite difference equations, which complicates the development of effective computational methods due to the inhomogeneity of the problem structures. Therefore, considering the discrete case facilitates the construction of computational methods [39] for the homogeneous-discrete case [45].

2. Stationary case. Determining the order of the fractional derivative α .

2.1. Algorithm for determining α using a discretized Voltaire integral equation of the second kind.

Let the oscillating system with liquid dampers (1.2) with the initial conditions is given

$$y(t_0) = 0, \quad y'(t_0) = y_{10}. \quad (2.1)$$

Thus, we get the Cauchy problem [40] (1.2)-(2.1).

Taking into account the definitions of the fractional Riemann-Liouville derivative [61], the subordinate term in (1.2) takes the form:

$$D^{\alpha} y(t) = \frac{d^2}{dt^2} \int_{t_0}^{t} \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!} y(\tau) d\tau, \quad t \geq t_0 > 0, \ \alpha \in (1,2). \qquad (2.2)$$

Then from (2.2) and integrating twice (1.2), we have:

$$y(t) + \int_{t_0}^{t} K_{\alpha}(t-\tau) y(\tau) d\tau = F(t), \ t \geq t_0 > 0, \ \alpha \in (1,2) \qquad (2.3)$$

$$K_{\alpha}(t-\tau) = \frac{a}{m} \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!} + \frac{b}{m}(t-\varepsilon), \quad (2.4)$$

$$F(t) = \frac{1}{m} \int_{t_0}^{t} (t-\tau) f(\tau) d\tau + y_{10}(t-t_0), \quad (2.5)$$

Now, discretizing the Volterra equations of the second kind (2.3) with a step $h = \frac{l - t_0}{n}$ in the interval $h = \frac{l - t_0}{n}$, we obtain:

$$y_i + \sum_{k=0}^{i-1} K_\alpha (t_i - t_k) y_k h = F_i, \quad i = \overline{1,n}, \text{(2.6)}$$

where

$$y_i = y(t_i), \; t_i = t_0 + ih, \; \overline{i = 1,n}, \; t_n = t_0 + nh = t_0 + n\frac{l - t_0}{n} = l, \quad \text{(2.7)}$$

$$K_\alpha (t_i - t_k) = \frac{a}{m}\frac{(t_i - t_k)^{1-\alpha}}{(1-\alpha)!} + \frac{b}{m}(t_i - t_k), \quad k = \overline{0, i-1}, \; i = \overline{1,n}, \quad \text{(2.8)}$$

$$F_i = \frac{1}{m}\sum_{k=0}^{i-1}(t_i - t_k) f(t_k)h + y_k (t_i - t_0), \quad i = \overline{1,n}. \quad \text{(2.9)}$$

Choosing α from $(0,1) \cup (1, 2)$ (the choice is given on the interval (1,2)) with a step $\frac{1}{p}$, we have:

$$\alpha_s = 1 + \frac{s}{p}, \; s = \overline{1, p-1}.$$

For the corresponding y_i from (2.6), we get:

$$y_i^s = F_i - h\sum_{m=0}^{i-1} K_{\alpha,s} (t_i - t_m) y_m^s, \quad s = \overline{1, p-1}, \; i = \overline{1,n}. \text{(2.10)}$$

To find the parameter α, we use the least squares method [23] and compose the following functional:

$$J = \sum_{s=1}^{p-1} (y_i - y_i^s)^2, \text{(2.11)}$$

where y_i depend on α (in the form (2.6)), and y_i^s are statistical data taking from practice. Then for determining α we get the equation:

$$\frac{\partial J}{\partial \alpha} = -2\sum_{s=1}^{p-1}(-\sum_{m=0}^{n-1} K_\alpha (t_n - t_m) y_m h + F_n - y_n^s)\sum_{m=0}^{n-1}\frac{\partial K_\alpha (t_n - t_m)}{\partial \alpha} y_m h = 0, \text{(2.12)}$$

more pimple form

$$
-2\sum_{s=1}^{p-1}\left(-\sum_{m=1}^{n-1}K_\alpha(t_n-t_m)y_m h+F_n-y_n^s\right)\times
$$

$$
\times\sum_{m=1}^{n-1}\frac{-(t_n-t_m)^{1-\alpha}\ln(t_n-t_m)\Gamma(2-\alpha)+(t_n-t_m)^{1-\alpha}\int_0^\infty e^{-t}t^{1-\alpha}\ln t dt}{\Gamma^2(2-\alpha)}y_m=0. \quad (2.13)
$$

where $\Gamma^2(2-\alpha)$ is the Euler function, having solved by a linear path of equation (2.13) with respect to α, we determine the desired fractional derivative.

Let us present the following algorithm for the solution of transcendental equation (2.13):

Algorithm:

1. ε - determining the accuracy of the solution of the problem and parameters a, b, F_i are set.
2. The segment $[a_1,\ a_2]$ is defined, where the root of the function $f(a)$ is sought, where $f(a_1)f(a_2)<0.$
3. N=10, i=4 are given.
4. The values of statistical data are from table1.
5. Are calculated the midpoint $\alpha=\frac{\alpha_1+\alpha_2}{2}$ and the value of $y(\alpha)$ according to the formula (2.12).
6. If $|y(\alpha)|<\varepsilon$ the process stops. Otherwise, if $y(\alpha_1)y(\alpha)<0$, we denote $\alpha_2=\alpha$ and if $y(\alpha)y(\alpha_2)<0$, then $\alpha_1=\alpha$.Go to step 5.

Note that by setting $n = \frac{\alpha_2 - \alpha_1}{N}$ we calculate $\alpha_i = \alpha_1 + (i-1)h$ for $i = \overline{1, N+1}$ and define the values of the function f whose graph is shown in Fig.2.

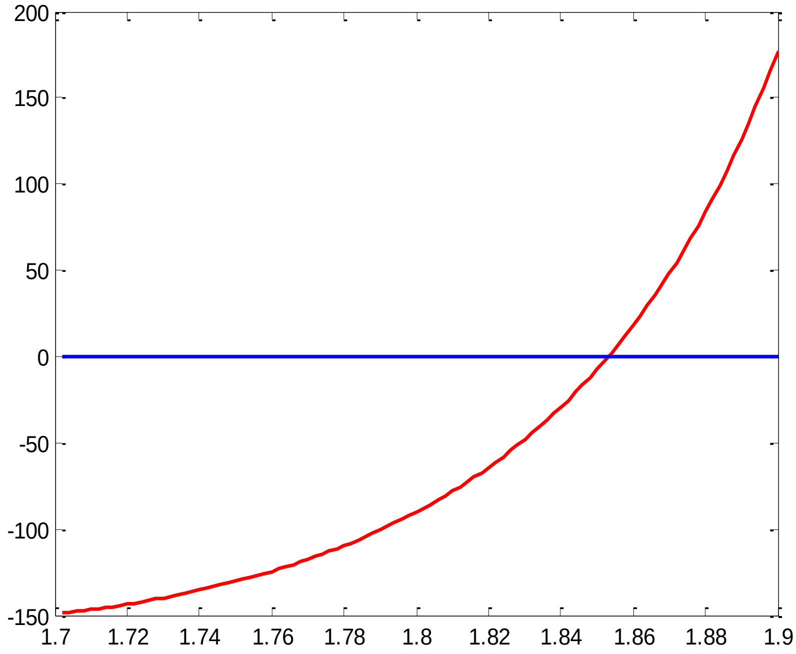

Fig.2. Dependency graph $y(\alpha)$ on α .

2.2.Asymptotic method. Finding the parameter α for a sufficiently large *m*.

Let m in (1.2) $m$is a sufficiently large, i.e.,taking $\varepsilon = \frac{1}{m}$ this equation can be written in the form:

$$\ddot{y} + \varepsilon a D^{\alpha} y + \varepsilon b y = \varepsilon f \qquad (2.14)$$

under condition (2.1), where its solution in the first approximation with respect to a small parameter ε has the form [31]:

$$y(t,\varepsilon) = y_{10}(t - t_0) + \varepsilon y_{10}\left[-\frac{a(t-t_0)^{3-\alpha}}{(3-\alpha)!} + \frac{5b(t-t_0)^3}{6} \right]. \qquad (2.15)$$

Note that in determining α for this case, it is necessary in (2.12) or (2.13) instead of y_m to take into account (2.15) at the point t_m. After corresponding transformations, we have the following equation

$$\frac{a(T-t_0)^{3-\alpha}}{\Gamma(4-\alpha)} = 5b\frac{(T-t_0)^3}{6} + \frac{1}{\varepsilon}\left[-(T-t_0) + \frac{\sum_{s=1}^{k} y_{Ts} y_{1s}}{\sum_{s=1}^{k} y_{1s}^2} \right], \qquad (2.16)$$

where $\Gamma(4-\alpha)$ is the Euler function, y_{Ts} is the final condition for the solution of the original system on the wake in the form (2.2) of the given statistical initial conditions y_{1s}:

$$\Gamma(4-\alpha)=\int_0^{\infty} e^{-t}t^{3-\alpha}dt.$$

It is easy to prove that [77] if

$$\frac{\ln(3-\alpha)+2+\dfrac{1}{2(3-\alpha)}}{\ln(T-t_0)}<1 \qquad (2.17)$$

is satisfied, then (2.16) has a unique solution.

Returning to (2.16), we represent it in the following form

$$\alpha=G(\alpha,T,\varepsilon,t_0), \qquad (2.18)$$

where

$$G(\cdot)=3-\frac{\ln F(t,\varepsilon,T)+\ln(3-\alpha)!}{\ln(T-t_0)}, \qquad (2.19)$$

$$F(t,\varepsilon,T)=5\frac{b}{a}\frac{(T-t_0)^3}{3!}+\frac{1}{a\varepsilon}\left[-(T-t_0)+\frac{\sum_{s=1}^{k} y_{Ts}y_{1s}}{\sum_{s=1}^{k} y_{1s}^2}\right]. \qquad (2.20)$$

Note that equation (2.18) with condition (2.17) has a unique solution.

If we choose the initial approximation α_0 in a close neighborhood of the solution $\overline{\alpha}$, we can prove that the sequence

$$\alpha_{k+1}=G(\alpha_k,T,\varepsilon,t_0)$$

converges to $\overline{\alpha}$, i.e. $\alpha_k\to\overline{\alpha}$.

3. Method for solving oscillatory systems, where fractional derivatives with a step of 1/q ($q\in N$) entering both the equation of motion and nonlocal boundary conditions.

3.1. The general case.

The general form of above equation (1.2) has the form [7,12,68]:

$$y''(t)+\sum_{k=0}^{2q-1} a_k D^{\frac{k}{q}} y(t)=f(t), \qquad 0<t_0<t<l, \qquad (3.1)$$

where $q\in N=\{1,2,3,\ldots\}$ - natural number, $a_k\in R$ - given real numbers, *f*(*t*) – continuous real valued function, in equation (3.1), the order of the derivative varies by 1/q to the second order.

For equation (3.1), we present the following nonlocal boundary conditions:

$$\sum_{k=0}^{2q-1}\left[\alpha_{jk} D^{\frac{k}{q}} y(t)\Big|_{t=t_0} + \beta_{jk} D^{\frac{k}{q}} y(t)\Big|_{t=l}\right] = \gamma_j, \qquad j=\overline{1,\ 2q}, \qquad (3.2)$$

where α_{jk}, β_{jk} and γ_j are given real numbers and boundary conditions (3.2) are linearly independent.

By means of substituting

$$D^{\frac{k}{q}} y(t) = z_k(t), \quad k=\overline{0,2q-1},$$

$$D^{\frac{2q}{q}} y(t) \equiv y''(t) = D^{\frac{1}{q}} z_{2q-1}(t), \qquad (3.3)$$

the boundary value problem (3.1), (3.2) is reduced to the form:

$$D^{\frac{1}{q}} z(t) = Az(t) + B(t), \qquad 0 < t_0 < t < l, \qquad (3.4)$$

$$\alpha z(t_0) + \beta z(l) = \gamma, \qquad (3.5)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 & -a_4 & . & . & . & . & -a_{2q-3} & -a_{2q-2} & -a_{2q-1} \end{pmatrix}, \qquad (3.6)$$

$$B(t) = (0 \quad 0 \quad 0 \quad 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad 0 \quad 0 \quad 0 \quad f(t))^T, \qquad 0 < t_0 < t < l, \qquad (3.7)$$

$$z(t) = (z_0(t) \quad z_1(t) \quad z_2(t) \quad \ldots \quad z_{2q-3}(t) \quad z_{2q-2}(t) \quad z_{2q-1}(t))^T, 0 < t_0 < t < l, \ (3.8)$$

$$\left\{\alpha = (\alpha_{jk})_{j=\overline{1,\ 2q},\, k=\overline{0,\ 2q-1}}; \quad \beta = (\beta_{jk})_{j=\overline{1,\ 2q},\, k=\overline{0,\ 2q-1}}\right. . \ (3.9)$$

Let

$$T = (t_{ij})_{i,j=1}^{2q} \ (3.10)$$

matrix transformation reduces A to a diagonal form, i.e.

$$AT = T\hat{A}, \qquad (3.11)$$

where

$$\hat{A} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{2q-1} & \\ 0 & & & & \lambda_{2q} \end{pmatrix}. \qquad (3.12)$$

Then with the replacement

$$Z(t) = TW(t), (3.13)$$

the homogeneous equation corresponding to (3.4) is reduced to the form:

$$D^{\frac{1}{q}}W(t) = \hat{A}W(t), \qquad (3.14)$$

the solution of which is given using the Mittag-Leffler function [4, 9, 27, 33, 37, 79, 81] has the following form:

$$W_k(t) = \sum_{m=0}^{\infty} \lambda_k^m \frac{t^{-1+\frac{m+1}{q}}}{\left(-1+\frac{m+1}{q}\right)!}, \quad k = \overline{1,2q}, \ (3.15)$$

where the elements λ_k of the diagonal matrix $\hat{A}$ are determined from the equation

$$\det(A - \lambda E) = |A - \lambda E| = 0, (3.16)$$

E - unit matrix of order $2q$. Thus, the matrix solution [24] of the homogeneous equation corresponding to (3.4) has the form:

$$Z(t) = \sum_{k=0}^{\infty} A^k \frac{t^{-1+\frac{k+1}{q}}}{\left(-1+\frac{k+1}{q}\right)!}. \qquad (3.17)$$

For the general solution of the homogeneous equation corresponding to (3.4) we obtain:

$$z(t) = Z(t)C = \sum_{k=0}^{\infty} A^k \frac{t^{-1+\frac{k+1}{q}}}{\left(-1+\frac{k+1}{q}\right)!} C, \qquad (3.18)$$

where C- arbitrary column with a size $2q$ with constant elements.

If

$$f(t) = 0, \qquad (3.19)$$

taking into account (3.18) in (3.5) we have:

$$\alpha Z(t_0)C + \beta Z(l)C = \gamma, (3.20)$$

or

$$C = [\alpha Z(t_0) + \beta Z(l)]^{-1}\gamma, (3.21)$$

if

$\det[\alpha Z(t_0)+\beta Z(l)]\neq 0.$ (3.22)

Then we get

Theorem 3.1. If a_k, $k=\overline{0,\ 2q-1}$- given real numbers, $f(t)=0$, α_{jk}, β_{jk}, γ_j for $j=\overline{1,\ 2q}$, $k=\overline{0,\ 2q-1}$. Also thegiven real numbers and the condition (3.2) are linearly independent, then the solution of the problem (3.4), (3.5) is given in the form (3.18), where C is defined in the form (3.21) under the condition (3.22).

The second method to solve the boundary value problem (3.4), (3.5). By integrating equations (3.4) of order 1/q, we reduce to the following Voltaire integral equation of the second kind [15, 36, 63]

$$z(t)=A\int_{t_0}^{t}\frac{(t-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}z(\tau)d\tau+F(t)+C\frac{t^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!},\ (3.23)$$

$\alpha z(t_0)+\beta z(l)=\gamma,$

where

$$F(t)=\int_{t_0}^{t}\frac{(t-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}B(\tau)d\tau. \qquad (3.24)$$

Substituting (3.23) into (3.5), we have

$$\alpha C\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}+\beta\left[A\int_{t_0}^{l}\frac{(l-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}z(\tau)d\tau+F(l)+C\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right]=\gamma, \qquad (3.25)$$

from which

$$C=\left(\alpha\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}+\beta\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right)^{-1}\left[\gamma-\beta A\int_{t_0}^{l}\frac{(l-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}z(\tau)d\tau+F(l)\right],\ (3.26)$$

with condition

$$\det\left(\alpha\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}+\beta\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right)\neq 0.\ (3.27)$$

Substituting (3.26) into (3.23), we arrive at an integral equation containing both Voltaire and Fredholm terms [59]. The theory of such equations was developed in the works [83]

3.2. Asymptotic method.

In the previous problem, each side (3.1) was divided into m (for large values of m) and assumed that $\frac{1}{m}=\varepsilon$ we have:

$$y''(x)+\sum_{k=0}^{2q-1}\varepsilon a_k D^{\frac{k}{q}}y(t)=\varepsilon f(t),\ \ 0<t_0<t<l, \quad (3.28)$$

$$\sum_{k=0}^{2q-1}\left[\alpha_{jk}D^{\frac{k}{q}}y(t)\Big|_{t=t_0}+\beta_{jk}D^{\frac{k}{q}}y(t)\Big|_{t=l}\right]=\gamma_j,\quad j=\overline{1,2q}. \qquad (3.29)$$

Similar to the previous one in section 3.1, we arrive at the following problem:

$$D^{\frac{1}{q}}Z(t,\varepsilon)=A(\varepsilon)Z(t,\varepsilon)+B(t,\varepsilon), \quad (3.30)$$

$$\alpha Z(t_0,\varepsilon)+\beta Z(l,\varepsilon)=\gamma, \qquad (3.31)$$

where

$$Z(t,\varepsilon)=(Z_0(t,\varepsilon)\ Z_1(t,\varepsilon)\ Z_2(t,\varepsilon)\ ...Z_{2q-1}(t,\varepsilon))^T, \quad (3.32)$$

$$A(\varepsilon)=A_0+\varepsilon A_1, \qquad (3.33)$$

$$B(t,\varepsilon)=\varepsilon l_{2q}f(t), \quad (3.34)$$

$\alpha,\ \beta,\ \gamma$ are given in the previous case.

A_0 - nilpotent of order $2q$, A_1-zero matrix in the last 2q*th* row $(-a_0-a_1\ -a_2\ ...\ -a_{2q-2}\ \ -a_{2q-1})$, $l_{2q}=(0\ 0\ 0...0\ 1)^T$ order$2q$.

$$A_0^{2q}=0, A_1^k=(-1)^{k-1}a_{2q-1}^{k-1}A_1. \qquad (3.35)$$

The solution of the system of differential equations (3.30) will be in the form:

$$Z(t,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k Z^{(k)}(t). \quad (3.36)$$

Substituting (3.36) into (3.30) taking into account (3.33)-(3.35)

$$D^{\frac{1}{q}}Z^{(0)}(t)=A_0Z^{(0)}(t), \qquad (3.37)$$

$$D^{\frac{1}{q}}Z^{(1)}(t)=A_0Z^{(1)}(t)+(A_1Z^{(0)}(t)+l_{2q}f(t)), \qquad (3.38)$$

$$D^{\frac{1}{q}}Z^{(s)}(t)=A_0Z^{(s)}(t)+A_1Z^{(s-1)}(t),\quad s\geq 2, \qquad (3.39)$$

with boundary conditions

$$\alpha Z^{(0)}(t_0) + \beta Z^{(0)}(l) = \gamma, \qquad (3.40)$$

$$\alpha Z^{(s)}(t_0) + \beta Z^{(s)}(l) = 0, \ \ s \geq 1. \quad (3.41)$$

Taking into account the condition (3.35) the matrix solution of system (3.37) has the form:

$$Z^0(t) = \sum_{k=0}^{2q-1} A_0^k \frac{t^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}, \qquad (3.42)$$

and the solution of boundary problem (3.37), (3.40) is found as follows:

$$z^{(0)}(t) = Z^{(0)}(t)\left[\alpha Z^{(0)}(t_0) + \beta Z^{(0)}(l)\right]^{-1}\gamma, \quad (3.43)$$

if

$$\det\left[\alpha Z^{(0)}(t_0) + \beta Z^{(0)}(l)\right] \neq 0. \quad (3.44)$$

The solution of homogeneous equation corresponding to equations (3.38) has the form:

$$z^{(1)}(t) = Z^{(0)}(t)C, \qquad (3.45)$$

where C-vector column of length $2q$ with arbitrary constant elements, and the solution of the boundary value problem (3.38), (3.41) for $s = 1$ has the form:

$$z^{(1)}(t) = -Z^{(0)}(t)[\alpha Z^{(0)}(t_0) + \beta Z^{(0)}(l)]^{-1}\left\{\beta\begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(l)\end{pmatrix} + \alpha\begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(t_0)\end{pmatrix}\right\} + \begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(t)\end{pmatrix}, \qquad (3.46)$$

where

$$\tilde{A}(t) = I^{\frac{1}{q}}\left\{f(t) + A_{1,2q}Z^{(0)}(t)[\alpha Z^{(0)}(t_0) + \beta Z^{(0)}(l)]^{-1}\gamma\right\}. \quad (3.47)$$

If we restrict with zero and first approximations, then for the solution of the boundary value problem (3.30), (3.31) we obtain:

$$z(t) = z^{(0)}(t) + \varepsilon z^{(1)}(t), \ (3.48)$$

where $z^{(0)}(t)$ and $z^{(1)}(t)$ are defined (3.43), (3.46) correspondingly.

Note that the results presented in item 3 can be used to construct program trajectories and control oscillatory systems with liquid dampers [21,25].

4. Construction of optimal regulators of stabilizing oscillatory systems with liquid dampers.

An algorithm is given for the design of optimal controllers for oscillatory systems using the method of liquid dampers by the analytical design of optimal Letov controllers [6,42] (this method being called the time method). Then one proceeds to the frequency method, i.e., the equation of motion by the Laplace transform and the Fourier transform, translates the quadratic functional into an algebraic equation, and then uses the elements of the calculus of variations to construct an optimal controller. This method is called Larin parametrization to construct optimal regulators [2,57,78] for linear quadratic control problems on an infinite time interval.

4.1. Construction of optimal regulators by the Letov's method [6,42].

Let the motion of the object described by the equation (3.1) has the following initial conditions

$$\left. D^{\frac{k}{q}} y(t) \right|_{t=t_0} = y_k(t_0),\ k = \overline{0, 2q-1}, (4.1)$$

where $y(t)$ - control action, $y_k(t_0)$ -given real numbers.

After corresponding transformations (3.1) is reduced to the normal system (3.4), i.e.

$$D^{\frac{1}{q}} z(t) = Az(t) + Gu(t), \qquad (4.2)$$

with initial conditions

$$z(t_0) = z_0 = (y_0(t_0),\ y_1(t_0),\ \dots,\ y_{2q-1}(t_0))^T,$$

where $z(\cdot)$, A, $u(t) = B(t)$ are defined from (3.6)-(3.8), $u(t)$ - we take it as a control action, $G = (0\ 0\ \dots\ 0\ 1)^T$.

The problem consist of finding the linear control law

$$U(t) = Kz(t), \qquad (4.3)$$

so that the closed system (4.2), (4.3).

$$D^{\frac{1}{q}} z(t) = (A + GK)z(t), \qquad (4.4)$$

be asymptotically stableandthe next quadratic functional

$$J = \int_0^\infty (z^0(t)Qz(t) + u'(t)Ru(t))dt \qquad (4.5)$$

received its minimum value along solutions (4.4).Here the sought constant matrix K with corresponding dimension, matrices $Q \geq 0$, $R > 0$ are given symmetric and also have the corresponding dimensions.

As shown in [2,6], the desired feedback target matrix K has the form:

$K = -R^{-1}\hat{T}_3\hat{T}_1^{-1},$ (4.6)

where

$\hat{T}_1^{-1}H\hat{T} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix},\ H = \begin{bmatrix} A & -GR^{-1}G' \\ -Q & -A' \end{bmatrix},\ T = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 \\ \hat{T}_3 & \hat{T}_4 \end{bmatrix}$

and H_+ with the dimension $q \times q$ has размерностиhas eigenvalues on the left half-plane, H_- –on the right. In this case, the closed system (4.1)

$D^{\alpha} z(t) = (A - GR^{-1}G'\hat{T}_3\hat{T}_1^{-1})z(t),\ z(t_0) = z_0$ (4.7)

has a solution

$z(t) = T_1\tilde{X}_1(t)C_1,$ (4.8)

Which for $t \to \infty,\ \tilde{X}_1(t) \to 0$ and therefore $z(t) \to 0$, where is defined in [2,6] :

$$\hat{X}_1(t) = \sum_{s=0}^{p-2} A_+^{\frac{s+p}{q}} e^{A_+^{\frac{p}{q}}t} \frac{1}{\Gamma(\frac{s+1}{p})} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} A_+^{\frac{pk}{q}} \frac{(t-t_0)^{\frac{s+1}{p}+k}}{\left(\frac{s+1}{p}+k\right)!} +$$

$$+ \sum_{s=0}^{p-2} A_+^{\frac{s}{q}} e^{A_+^{\frac{p}{q}}t_0} \frac{t^{\frac{s+1}{p}-1}}{\Gamma(\frac{s+1}{p})} + A_+^{\frac{p-1}{q}} e^{A_+^{\frac{p}{q}}t}, \qquad (4.9)$$

$C_1$is an arbitrary constant vector defining from the initial condition $z(t_0) = z_0$.

4.2. Larin parametrization. Consider a simple form of the equation (3.1), i.e. let

$my''(t) + aD^{\alpha} y(t) + by(t) = u(t),\ \ \alpha = \frac{p}{q},$ (4.10)

where α is one of $\frac{k}{q}$ in (3.1) and has an initial condition

$y(t_0) = 0,\ \ y'(t_0) = y_1.$ (4.11)

The problem consists of finding

$u = Kt$ (4.12)

that the functional

$J = \frac{1}{2}\int_0^{\infty} (ry^2 + cu^2)dt$ (4.13)

obtains a minimum value and the closed system (4.10), (4.12) is asymptotically stable.

Taking the Laplace transform to (4.10), we have

$P(s)\tilde{y}(s) = M(s)\tilde{u}(s) + \Psi(s),$ (4.14)

where

$P(s) = mS^2 + aS^{\frac{p}{q}} + b,\ \ M(s) = 1,\ \ \Psi(s) = y.$ (4.15)

Using Fourier transforms and Parseval's identity [69], we write functional (4.13) in the form

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(r\,\tilde{y}(s)\tilde{y}(-s) + c\tilde{u}(s)\tilde{u}(-s))ds. \quad (4.16)$$

In a comprehensive area, it is necessary to find a law on regulation

$$\omega_0(s)\tilde{u}(s) = \omega_1(s)x(s), (4.17)$$

so that the functional (4.16) receives the minimum value, and the closed system (4.14) + (4.17) is asymptotically stable.

Composing the matrix Z from [49, 70, 72, 76]

$$Z = \begin{bmatrix} P(s) & -M(s) \\ A(s) & B(s) \end{bmatrix}, \qquad (4.18)$$

we select freely varying parameters *A (s), B (s)* so that $Z^{-1}(s)$ is analytically on the right half-plane, i.e. $\det Z(s) = P(s)B(s) + M(s)A(s)$ should be Hurwitz or constantly. In this case

$$A(s) = 1, \quad B(s) = 0. \quad (4.19)$$

Using Larin parametrization [70,71] and accepting $\omega(s) = \frac{\omega_1(s)}{\omega_0(s)}$ we get

$$\omega(s) = \frac{\Phi(s)P(s) - 1}{\Phi(s)}, (4.20)$$

where $\Phi(s)$ is Larin parameter [72] physically realizable - analytically on the right half-plane

$$\Phi(s) = \frac{B_0(s)}{D(s)}, (4.21)$$

where $B_0(s) + B_+(s) = \frac{T(s)}{D(-s)}$, $D_+(s)D_-(s) = (r + cP(s)P(-s))K\Psi_1(s)$.

$$T(s) = -CP(-s)\Psi_1^2. (4.22)$$

Here B_0 is the integer part, the fractional part $B_-(s)$ has poles in the right half-plane after separating the expressions $\frac{T(s)}{D(-s)}$**.** In case of factorization, $D(s)$ has zeros on the left half-plane from (4.22).

Substituting $\Phi(s)$ from (4.21) into (4.20), then for $\omega(s)$ we get:

$$\omega(s) = \frac{B_0(s)P(s) + D(s)}{B_0(s)}, \quad (4.23)$$

i.e.

$$\omega_0(s) = B_0(s),\ \omega_1(s) = B_0(s)P(s) + D(s). \quad (4.24)$$

Let’s prove that the closed system (4.14)+(4.17) is asymptotically stable. Substituting $\omega_0(s),\ \omega_1(s)$ from (4.23), (4.24) into

$$\det\begin{bmatrix} P(s) & -M(s) \\ \omega_1(s) & \omega_0(s) \end{bmatrix} = D(s), \quad (4.25)$$

we obtain that the closed system is asymptotically stable.

Another parameterization, the so-called Youla-Kucera-Degoer [73-75], unlike [70-73], suggest choosing $\alpha(s),\ \beta(s)$ will satisfy the following

$P(s)\beta(s)+M\alpha(s)=1.$

Diafant equation which is a special case of Larin's parametrization [70,72,76].

4.3 Asymptotic method. For a sufficiently large $m>>1$ каквпункте 3.2 применяем $\varepsilon=\dfrac{1}{m}$ and we have equation (3.27) which appears ε, where, as in (3.2), (3.3), and (4.2), we get

$D^{\frac{1}{q}}z(t)=(A_0+\varepsilon A_1)z(t)+\varepsilon Gu(t).$ (4.26)

Then looking for

$$T=T^{(0)}+\varepsilon T^{(1)}=\begin{bmatrix}\hat{T}_1^{(0)}+\varepsilon\hat{T}_1^{(1)} & \hat{T}_2^{(0)}+\varepsilon\hat{T}_2^{(1)}\\ \hat{T}_3^{(0)}+\varepsilon\hat{T}_3^{(1)} & \hat{T}_4^{(0)}+\varepsilon\hat{T}_4^{(1)}\end{bmatrix},\ \ R=\frac{1}{\varepsilon}\hat{R}.$$

For K from (4.6) we have the following expression

$K=-\hat{R}^{-1}G'[\hat{T}_3^{(0)}\hat{T}_1^{(0)^{-1}}+\varepsilon(-\hat{T}_3^{(0)}T_1^{(0)^{-1}}T_1^{(1)}T_1^{(0)^{-1}}+T_3^{(1)}T_1^{(0)^{-1}})],$ (4.27)

where $\hat{T}_3^{(0)},\hat{T}_1^{(0)},T_1^{(1)},T_3^{(1)}$ are defined from [ ].

Now, using a particular method, we can redistribute $\omega_0(s),\ \omega_1(s)$ (4.17), when equation (4.10) for a sufficiently large $m>>1$ goes to the form

$\ddot{y}+\varepsilon aD^{\alpha}y+\varepsilon by=\varepsilon u(t)$

and for the given case

$P=s^2+\varepsilon as^{\frac{p}{q}}+\varepsilon b=P_0(s)+\varepsilon P_1(s),\ M=\varepsilon,\ \psi=y_1,$

where $P_0(s)=s^2,\ P_1(s)=as^{\frac{p}{q}}+b,$

and as a result of this, $D(s)D(-s)$ at the first approximation with respect to the small parameter ε has the form

$D(s)D(-s)\approx c\psi_1^2s^4-2ac\varepsilon s^{2+\frac{p}{q}}-2\psi_1^2b\varepsilon s^2+r$ (4.28)

and further

$T(s)=-c(s^2-\varepsilon as^{\frac{p}{q}}+\varepsilon b)\psi_1^2.$ (4.22`)

Similarly to the previous case, it is possible to determine $\omega_0(s),\ \omega_1(s)$ at the first approximation with respect to a small parameter ε.

$$D(\rho)D(-\rho)=c\psi_1^2(\rho^{2q}+O\rho^{2q-1}+...+O\rho^{q+k+1}-\frac{2a}{\psi_1^2}\varepsilon\rho^{q+k}+O\rho^{q+k-1}+...+O\rho^{q+1}-$$
$$-\frac{2b}{c}\varepsilon\rho^{q}+O\rho^{q-1}+...+O\rho+\frac{r}{c\psi_1^2}). \quad (4.29)$$

Now we factorize [52-54] the polynomial (4.29) in the following form

$D(\rho)=\sqrt{c}\psi_1(\rho^q+(L'+G'\Pi)N),$ (4.30)

where in this case

$$L = 0,\quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},\ N(\rho) = \begin{bmatrix} 1 \\ \rho \\ \vdots \\ \rho^{q-1} \end{bmatrix},$$

and the matrix Π is a positive definite solution to the following matrix algebraic Riccati equation [46, 50, 55, 56, 65,67]

$\Pi F + F'\Pi - (\Pi G + L)(G'\Pi + L') + R = 0.$ (4.31)

Here $F,\ R$ are block two and three-diagonal matrices as

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},\quad R = R_0 + R_1',$$

$R_0 = diag[r,0,...,0,-2ac\varepsilon,0,...,0],$

$R_1 = diag\left[0,...,0,\frac{b}{c}q,...,0\right],\ F,\ R = \tilde{R}_0 + \varepsilon\tilde{R}_1,\ \tilde{R}_0 = diag[r,0,...,0],\ \tilde{R}_1 = diag\left[0,...,-rac,0,...,\frac{b}{c},0,...,0\right]$

Choosing such solution Π, the equation (4.31) so that the matrices

$F - G(L' + G'\Pi)$ (4.32)

are Hurwitz, i.e.. the eigenvalues of (4.32) lie in the left-half plane.

Now we are looking for (4.31) in the form

$\Pi = \Pi_0 + \varepsilon\Pi_1.$

For $\Pi_0,\ \Pi_1$ we have the following

$\Pi_0 F + F'\Pi_0 - \Pi_0 GG'\Pi_0 + \tilde{R}_0 = 0,$ (4.33)

$\Pi_1(F - GG'\Pi_0) + (F' - \Pi_0 GG')\Pi_1 + \tilde{R}_1 = 0,$ (4.34)

i.e.solving Riccati equation (4.33) with respect to Π_0 we find a solution so that the matrix $F - GG'\Pi_0$ is Hurwitz. Further solving the Lyapunov equation (4.34) [48] we get $\Pi_1 \geq 0$ and we restore the factorized polynomial [62,66] $D(s)$ in the following form

$D(s) = D_0(s) + \varepsilon D_1(s),$ (4.35)

where

$D_0(s) = \sqrt{c}\psi_1(s^2 + G'\Pi_0 N(s)),\ D_1(s) = \sqrt{c}\psi_1 G'\Pi N(s),$

where $N'(s) = \left[1,\ s^{\frac{2}{q}},...,s^{2(1-\frac{1}{q})}\right]'.$

We can easily show that in this case from (4.22) has the representation

$T(s) = -\psi_1^2 cs^2 + \varepsilon(as^{\frac{p}{q}} - b)c\psi_1^2.$ (4.36)

Then from (4.22) $B_0(s) = -\psi_1\sqrt{c}$ and therefor from (4.24)

$\omega_0(s) = -\psi_1\sqrt{c},$ (4.37)

$\omega_1(s) = (-\psi_1\sqrt{c}P_0(s) + D_0(s)) + \varepsilon(-\psi_1\sqrt{c}P_1(s) + D_1(s)).$

Now we can easily prove that the corresponding closed system is asymptotically stable equal to $D_0(s) + \varepsilon D_1(s)$ which is equal in the first approximation to Hurwitz polynomial $D(s)$.

5. Discretization problem (1.2), (2.1).

5.1.Reduction the Cauchy problem (1.2), (2.1) to an integral equation of the second kind with respect to y``.

We consider the following initial problem for a second-order ordinary linear differential equation with constant coefficients and fractional derivatives [10, 16, 34, 35, 60, 80, 82] in subordinate terms, i.e.

$$y''(x) + aD^\alpha y(x) + by(x) = f(x), \quad x > 0, \ \alpha \in (1,2), \qquad (5.1)$$

$$\begin{cases} y(0) = 0, \\ y'(0) = y_{10}, \end{cases} \qquad (5.2)$$

where the coefficients *a*, *b* of the equation (5.1) and initial data y_{10} – given real numbers, the right-hand part of equation (5.1) – given continuous real-valued function, $y(x)$- the required function.

The initial problem (5.1) - (5.2) is reduced to the integral Voltaire equation of the second kind with respect to $y''(x)$.

Let's perform the following transformation:

$$D^\alpha y(x) = D^{\alpha-1}Dy(x) = DD^{\alpha-1}y(x), \qquad (5.3)$$

where $D = \dfrac{d}{dx}, \ \alpha - 1 \in (0,1).$

Then

$$D^{\alpha-1}y(x) = D\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y(t)dt, \quad (5.4)$$

and

$$D^\alpha y(x) = D^2\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y(t)dt. \quad (5.5)$$

Now we transform the integral in (5.5) so that the differentiated ones can be introduced under the integral sign:

$$\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y(t)dt = -\int_0^x y(t)d_t\frac{(x-t)^{2-\alpha}}{(2-\alpha)!}, \quad (5.6)$$

considering $2-\alpha > 0$, we integrate the integral in (5.6) by parts:

$$-\int_0^x y(t)d_t\frac{(x-t)^{2-\alpha}}{(2-\alpha)!} = -\frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y(t)\Big|_{t=0}^{x} + \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y'(t)dt = \frac{x^{2-\alpha}}{(2-\alpha)!}y(0) +$$
$$+\int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y'(t)dt = \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y'(t)dt, \qquad (5.7)$$

for obtaining (5.7), we take into account the first condition from (5.2). Then from (5.4) we get:

$$D^{\alpha-1}y(x) = D\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y(t)dt = D\int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y'(t)dt = \int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y'(t)dt. \quad (5.8)$$

In the obtained integral (5.8) we will carry out the above operation again, i.e.

$$\begin{aligned}&\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y'(t)dt = -\int_0^x y'(t)d_t\frac{(x-t)^{2-\alpha}}{(2-\alpha)!} = -\frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y'(t)\Big|_{t=0}^{x} + \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y''(t)dt = \\ &= \frac{x^{2-\alpha}}{(2-\alpha)!}y'(0) + \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y''(t)dt = \frac{x^{2-\alpha}}{(2-\alpha)!}y_{10} + \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y''(t)dt.\end{aligned} \quad (5.9)$$

Thus from (5.5) we get:

$$\begin{aligned}&D^{\alpha}y(x) = DD^{\alpha-1}y(x) = D\left[\frac{x^{2-\alpha}}{(2-\alpha)!}y_{10} + \int_0^x \frac{(x-t)^{2-\alpha}}{(2-\alpha)!}y''(t)dt\right] = \\ &= \frac{x^{1-\alpha}}{(1-\alpha)!}y_{10} + \int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y''(t)dt.\end{aligned} \quad (5.10)$$

Now we consider the third term on the left-hand side of (5.1):

$$\begin{aligned}&y(x) = \int_0^x y'(t)dt + y(0) = \int_0^x y'(t)dt = \int_0^x dt\left\{\int_0^t y''(\tau)d\tau + y'(0)\right\} = \int_0^x dt\int_0^t y''(\tau)d\tau + \int_0^x y_{10}dt = \\ &= \int_0^x y''(\tau)d\tau\int_\tau^x dt + y_{10}x = \int_0^x (x-t)y''(t)dt + y_{10}x.\end{aligned} \quad (5.11)$$

Now considering (5.10) and (5.11), the equation (5.1) take the form:

$$y''(x) + a\left[\frac{x^{1-\alpha}}{(1-\alpha)!}y_{10} + \int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y''(t)dt\right] + b\left[\int_0^x (x-t)y''(t)dt + y_{10}x\right] = f(x),$$

or

$$y''(x) + a\int_0^x \frac{(x-t)^{1-\alpha}}{(1-\alpha)!}y''(t)dt + b\int_0^x (x-t)y''(t)dt = f(x) - a\frac{x^{1-\alpha}}{(1-\alpha)!}y_{10} - by_{10}x,$$

or

$$y''(x) + \int_0^x K_\alpha(x-t)y''(t)dt = F(x),\ x > 0, \quad (5.12)$$

where

$$K_\alpha(x-t) = a\frac{(x-t)^{1-\alpha}}{(1-\alpha)!} + b(x-t), \qquad (5.13)$$

$$F(x) = f(x) - \left[a \frac{x^{1-\alpha}}{(1-\alpha)!} + bx \right] y_{10}. \qquad (5.14)$$

Here we will discretize the Voltaire integral equation of the second kind (5.12).

5.2. Discretization.

Considering $y(x) = y_n$, $F(x) = F_n$ and

$$y''(x) \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}, \qquad (5.15)$$

we obtain:

$$\frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} + h\sum_{k=0}^{n-1} K_\alpha (x_n - x_k) \frac{y_{k+2} - 2y_{k+1} + y_k}{h^2} = F_n, \; n \geq 0$$

where$h$-step, $x = x_n$, $x_0 = 0$, $t = x_k = kh$.

Then the obtained system of algebraic equations will take the form [51]:

$$y_{n+2} = 2y_{n+1} - y_n - h\sum_{k=0}^{n-1} \left[a \frac{(x_n - x_k)^{1-\alpha}}{(1-\alpha)!} + b(x_n - x_k) \right] (y_{k+2} - 2y_{k+1} + y_k) +$$
$$+ h^2 \left\{ f_n - \left[a \frac{x_n^{1-\alpha}}{(1-\alpha)!} + bx_n \right] y_{10} \right\}, \quad n \geq 0, \quad f_n = f(x_n), f_0 = f(0). \qquad (5.16)$$

5.2.1. The case of even indices.

Representing (5.16) for an even and an odd indices, we obtain the following pair of relations:

$$y_{2m} = 2y_{2m-1} - y_{2m-2} - h\sum_{k=0}^{2m-3} \left[a \frac{(x_{2m-2} - x_k)^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_k) \right] (y_{k+2} - 2y_{k+1} + y_k) +$$
$$+ h^2 \left\{ f_{2m-2} - \left[a \frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!} + bx_{2m-2} \right] y_{10} \right\}, \quad m \geq 1,$$

or

$$y_{2m}=2y_{2m-1}-y_{2m-2}-h\sum_{k=0}^{2m-4}\left[a\frac{(x_{2m-2}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_k)\right](y_{k+2}-2y_{k+1}+y_k)-$$

$$-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]y_{2m-1}+$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]y_{2m-2}-$$

$$-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]y_{2m-3}+$$

$$+h^2\left\{f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right\},\quad m\ge 1,$$

or

$$y_{2m}=\left\{2-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\right\}y_{2m-1}+$$

$$+\left\{-1-h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]+\right.$$

$$\left.+2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\right\}y_{2m-2}+$$

$$+\left\{-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]-\right.$$

$$-h\left[a\frac{(x_{2m-2}-x_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-5})\right]+$$

$$\left.+2h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]\right\}y_{2m-3}-$$

$$-h\sum_{k=0}^{2m-6}\left[a\frac{(x_{2m-2}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_k)\right]y_{k+2}+$$

$$+2h\sum_{k=0}^{2m-5}\left[a\frac{(x_{2m-2}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_k)\right]y_{k+1}-$$

$$-h\sum_{k=0}^{2m-4}\left[a\frac{(x_{2m-2}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_k)\right]y_k+$$

$$+h^2\left\{f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right\}.$$

Finally, we group the resulting expression as follows:

$$
\begin{aligned}
y_{2m} = & \left\{ 2 - h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-3}) \right] \right\} y_{2m-1} + \qquad (5.17)\\
& + \left\{ -1 - h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-4}) \right] + \right.\\
& \left. + 2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-3}) \right] \right\} y_{2m-2} +\\
& + \left\{ - h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-3}) \right] - \right.\\
& - h\left[a\frac{(x_{2m-2}-x_{2m-5})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-5}) \right] +\\
& \left. + 2h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{2m-4}) \right] \right\} y_{2m-3} +\\
& + \sum_{k=2}^{2m-4} \left\{ - h\left[a\frac{(x_{2m-2}-x_k)^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_k) \right] + \right.\\
& + 2h\left[a\frac{(x_{2m-2}-x_{k-1})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{k-1}) \right] -\\
& \left. - h\left[a\frac{(x_{2m-2}-x_{k-2})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_{k-2}) \right] \right\} y_k +\\
& + \left\{ 2h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_0) \right] \right.\\
& \left. - h\left[a\frac{(x_{2m-2}-x_1)^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_1) \right] \right\} y_1 +\\
& + \left\{ - h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2}-x_0) \right] \right\} y_0 +\\
& + h^2 \left\{ f_{2m-2} - \left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!} + b x_{2m-2} \right] y_{10} \right\}, \quad m \ge 1.
\end{aligned}
$$

Thus (5.17) is a discrete version of problem (1.2), (2.1) when the index is even.

5.2.2. Odd indices.

Now we consider the case when the indices are odd, i.e.

$$y_{2m+1}=2y_{2m}-y_{2m-1}-h\sum_{k=0}^{2m-2}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right](y_{k+2}-2y_{k+1}+y_k)+$$
$$+h^2\left\{f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right\},\quad m\geq 1,$$

or

$$y_{2m+1}=2y_{2m}-y_{2m-1}-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]y_{2m}-$$
$$-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]y_{2m-1}-$$
$$-h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]y_{2m-2}-$$
$$-h\sum_{k=0}^{2m-5}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_{k+2}+$$
$$+2h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]y_{2m-1}+$$
$$+2h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]y_{2m-2}+$$
$$+2h\sum_{k=0}^{2m-4}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_{k+1}-$$
$$-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]y_{2m-2}-$$
$$-h\sum_{k=0}^{2m-3}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_k+$$
$$+h^2\left\{f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right\},$$

or

$$y_{2m+1}=\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}y_{2m}+$$

$$+\left\{-1-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]+\right.$$

$$\left.+2h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}y_{2m-1}+$$

$$+\left\{-h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]-\right.$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]+$$

$$\left.+2h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]\right\}y_{2m-2}-$$

$$-h\sum_{k=0}^{2m-5}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_{k+2}+$$

$$+2h\sum_{k=0}^{2m-4}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_{k+1}-$$

$$-h\sum_{k=0}^{2m-3}\left[a\frac{(x_{2m-1}-x_k)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_k)\right]y_k+$$

$$+h^2\left\{f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right\},\ m\geq 1.$$

Similarly to (5.17), we group the resulting expression as follows:

$$y_{2m+1}=\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}y_{2m}+$$

$$+\left\{-1-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]+\right. \qquad (5.18)$$

$$\left.+2h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}y_{2m-1}+$$

$$+\left\{-h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]-\right.$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]+$$

$$\left.+2h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]\right\}y_{2m-2}+$$

$$+\sum_{k=2}^{2m-3}\left\{-h\left[a\frac{(x_{2m-1}-x_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-2})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-1}-x_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-1})\right]-$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_{k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k})\right]\right\}y_k+$$

$$+\left\{2h\left[a\frac{(x_{2m-1}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{0})\right]-\right.$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_{1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{1})\right]\right\}y_1+$$

$$+\left\{-h\left[a\frac{(x_{2m-1}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{0})\right]\right\}y_0+$$

$$+h^2\left\{f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right\},\ \ m\geq 1.$$

Finally, taking into account (5.17) in (5.18), we represent it in the following form:

$$y_{2m+1}=\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left\{2-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\right\}y_{2m-1}+$$

$$+\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left\{-1-h\left[a\frac{(x_{2m--2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]+\right.$$

$$\left.+2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\right\}y_{2m-2}+$$

$$+\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left\{-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]-\right.$$

$$-h\left[a\frac{(x_{2m-2}-x_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-5})\right]+$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]\Bigg\}y_{2m-3}+$$

$$+\sum_{k=2}^{2m-4}\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\Bigg\{-h\left[a\frac{(x_{2m-2}-x_{k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k})\right]+$$

$$+2h\left[a\frac{(x_{2m-2}-x_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k-1})\right]-$$

$$-h\left[a\frac{(x_{2m-2}-x_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k-2})\right]\Bigg\}y_{k}+$$

$$+\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\Bigg\{2h\left[a\frac{(x_{2m-2}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{0})\right]-$$

$$-h\left[a\frac{(x_{2m-2}-x_{1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{1})\right]\Bigg\}y_{1}+$$

$$+\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left\{-h\left[a\frac{(x_{2m-2}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{0})\right]\right\}y_{0}+$$

$$+h^{2}\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left\{f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right\}+$$

$$+\Bigg\{-1-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\Bigg\}y_{2m-1}+$$

$$+\left\{-h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]-$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}y_{2m-2}+$$

$$+\sum_{k=2}^{2m-3}\left\{-h\left[a\frac{(x_{2m-1}-x_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-2})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-1}-x_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-1})\right]-$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_{k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k})\right]\right\}y_k+$$

$$+\left\{2h\left[a\frac{(x_{2m-1}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{0})\right]-\right.$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_{1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{1})\right]\right\}y_1+$$

$$+\left\{-h\left[a\frac{(x_{2m-1}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{0})\right]\right\}y_0+$$

$$+h^2\left\{f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right\},\ m\geq 1,$$

or by grouping the corresponding terms, we have:

$$y_{2m+1}=\left\{\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times\right. \qquad (5.19)$$

$$\times\left(2-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\right)-$$

$$-1-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\Bigg\}y_{2m-1}+$$

$$+\Bigg\{\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times$$

$$\times\Bigg(-1-h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]+$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]\Bigg)-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\Bigg\}y_{2m-2}+$$

$$+\Bigg\{\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times$$

$$\Bigg(-h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right]-$$

$$-h\left[a\frac{(x_{2m-2}-x_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-5})\right]+$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]\Bigg)-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-5})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-4})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-3})\right]\Bigg\}y_{2m-3}+$$

$$+\sum_{k=2}^{2m-4}\left\{2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right\}\times$$

$$\times\left(-h\left[a\frac{(x_{2m-2}-x_{k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-2}-x_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k-1})\right]-$$

$$\left.-h\left[a\frac{(x_{2m-2}-x_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{k-2})\right]\right)-$$

$$-h\left[a\frac{(x_{2m-1}-x_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-2})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k-1})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{k})\right]\Bigg\}y_{k}+$$

$$+\left\{\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times\right.$$

$$\times\left(2h\left[a\frac{(x_{2m-2}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{0})\right]-\right.$$

$$\left.-h\left[a\frac{(x_{2m-2}-x_{1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{1})\right]\right)+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{0})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{0})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{1})\right]\Bigg\}y_{1}+$$

$$+\left\{\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times\right.$$

$$-\left(h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_0)\right]\right)-$$

$$\left.-h\left[a\frac{(x_{2m-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_0)\right]\right\}y_0+$$

$$+h^2\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times$$

$$\times\left(f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right)+$$

$$\left.+h^2\left(f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right)\right\},\ m\geq 1.$$

Thus, (5.19) is an odd variant of the discretization of the Cauchy problem (1.2), (2.1).

5.3. Combining the discretization of even and odd cases as a system.

We take the following notation:

$$W_k=\begin{pmatrix} y_{2k} \\ y_{2k+1}\end{pmatrix},\ k\geq 0,\quad (5.20)$$

Then combining (5.17) and (5.19), taking into account (5.20), we obtain the following representation for the system of algebraic equations:

$$W_m=\sum_{k=0}^{m-1}A^{(k)}W_k+F_m,\ \ m\geq 1,\quad (5.21)$$

where

$$A^{(k)}=\begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)}\end{pmatrix},\ \ k\geq 0,\quad F_m=\begin{pmatrix} F_{m1} \\ F_{m2}\end{pmatrix},\ (5.22)$$

$$A_{11}^{(m-1)}=-1-h\left[a\frac{(x_{2m-2}-x_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-4})\right]+\ (5.23_1)$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2m-3})\right],$$

$$A_{12}^{(m-1)} = 2 - h\left[a\frac{(x_{2m-2} - x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2m-3})\right], \quad (5.23_2)$$

$$A_{21}^{(m-1)} = \left(2 - h\left[a\frac{(x_{2m-1} - x_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-2})\right]\right)\times \quad (5.23_3)$$

$$\times\left(-1 - h\left[a\frac{(x_{2m-2} - x_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2m-4})\right] +\right.$$

$$\left.+ 2h\left[a\frac{(x_{2m-2} - x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2m-3})\right]\right) -$$

$$- h\left[a\frac{(x_{2m-1} - x_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-4})\right] +$$

$$+ 2h\left[a\frac{(x_{2m-1} - x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-3})\right] -$$

$$- h\left[a\frac{(x_{2m-1} - x_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-2})\right],$$

$$A_{22}^{(m-1)} = \left(2 - h\left[a\frac{(x_{2m-1} - x_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-2})\right]\right)\times (5.23_4)$$

$$\times\left(2 - h\left[a\frac{(x_{2m-2} - x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2m-3})\right]\right) -$$

$$-1 - h\left[a\frac{(x_{2m-1} - x_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-3})\right] +$$

$$+ 2h\left[a\frac{(x_{2m-1} - x_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-1} - x_{2m-2})\right]$$

$$A_{11}^{(k)} = -h\left[a\frac{(x_{2m-2} - x_{2k})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2k})\right] + (5.24_1)$$

$$+ 2h\left[a\frac{(x_{2m-2} - x_{2k-1})^{1-\alpha}}{(1-\alpha)!} + b(x_{2m-2} - x_{2k-1})\right] -$$

$$-h\left[a\frac{(x_{2m-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k-2})\right],\ k=\overline{1,m-2},$$

$$A_{12}^{(k)}=-h\left[a\frac{(x_{2m-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k+1})\right]+ \qquad (5.24_2)$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k})\right]-$$

$$-h\left[a\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k-1})\right],\ \ k=\overline{1,m-2},$$

$$A_{21}^{(k)}=\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times \qquad (5.24_3)$$

$$\times\left(-h\left[a\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k-1})\right]-$$

$$\left.-h\left[a\frac{(x_{2m-2}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k-2})\right]\right)-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k-2})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k-1})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k})\right],\ k=\overline{1,m-2},$$

$$A_{22}^{(k)}=\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times \qquad (5.24_4)$$

$$\times\left(-h\left[a\frac{(x_{2m-2}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k+1})\right]+\right.$$

$$+2h\left[a\frac{(x_{2m-2}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_{2k})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\Bigg)-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k-1})\right]+$$

$$+2h\left[a\frac{(x_{2m-1}-x_{2k})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k})\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2k+1})\right],\quad k=\overline{1,m-2},$$

$$A_{11}^{(0)}=-h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_0)\right],\ (5.25_1)$$

$$A_{12}^{(0)}=2h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_0)\right]-\ (5.25_2)$$

$$-h\left[a\frac{(x_{2m-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_1)\right],$$

$$A_{21}^{(0)}=\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times\ (5.25_3)$$

$$\times\left(-h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_0)\right]\right)-$$

$$-h\left[a\frac{(x_{2m-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_0)\right],$$

$$A_{22}^{(0)}=\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times\ (5.25_4)$$

$$\times\left(2h\left[a\frac{(x_{2m-2}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_0)\right]-\right.$$

$$\left.-h\left[a\frac{(x_{2m-2}-x_1)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-2}-x_1)\right]\right)+$$

$$2h\left[a\frac{(x_{2m-1}-x_0)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_0)\right]-$$

$$-h\left[a\frac{(x_{2m-1}-x_1)^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_1)\right],$$

$$F_{m1}=h^2\left\{f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right\},\ m\geq 1.\ (5.26_1)$$

$$F_{m2}=h^2\left(2-h\left[a\frac{(x_{2m-1}-x_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(x_{2m-1}-x_{2m-2})\right]\right)\times (5.26_2)$$

$$\times\left(f_{2m-2}-\left[a\frac{x_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-2}\right]y_{10}\right)+$$

$$+h^2\left(f_{2m-1}-\left[a\frac{x_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bx_{2m-1}\right]y_{10}\right),\ m\geq 1.$$

5.4. A classical analogue of the discretization of problem (1.2), (2.1).

Now (5.21) can be represented in the form:

$$W_m=A^{(m-1)}W_{m-1}+\sum_{k=0}^{m-2}A^{(k)}W_k+F_m,\quad m\geq 1.\ (5.27)$$

Expressing the sum in (5.27) all terms through W_0, we have:

$$W_m=A^{(m-1)}W_{m-1}+\left[1+\sum_{j=1}^{m-2}\sum_{m-2\geq i_j>i_{j-1}>\ldots>i_2>i_1\geq 1}\prod_{k=1}^{j}A^{(i_k)}\right]A^{(0)}W_0+(5.28)$$

$$+\sum_{s=2}^{m-1}\left[1+\sum_{j=s}^{m-2}\sum_{m-2\geq i_j>i_{j-1}>\ldots>i_s\geq s}\prod_{k=s}^{j}A^{(i_k)}\right]A^{(s-1)}F_{s-1}+F_m,\quad m\geq 2,$$

$$W_1=A^{(0)}W_0+F_1.$$

Finally, solving (5.28), i.e. excluding also W_{m-1}, we get:

$$W_m=\left[1+\sum_{j=1}^{m-1}\sum_{m-2\geq i_j>i_{j-1}>\ldots>i_2>i_1\geq 1}\prod_{k=1}^{j}A^{(i_k)}\right]A^{(0)}W_0+\quad (5.29)$$

$$+\sum_{s=2}^{m}\left[1+\sum_{j=s}^{m-1}\sum_{m-1\geq i_j>i_{j-1}>\ldots>i_s\geq s}\prod_{k=s}^{j}A^{(i_k)}\right]A^{(s-1)}F_{s-1}+F_m,\quad m\geq 1.$$

Thus (5.27) is a discretization of the Cauchy problem (1.2), (2.1) with a difference of one time step between (5.28).

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